Generative subgrid-scale modeling

Jiaxi Zhao
joint with S. Arisaka & Q. Li @ NUS

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Data-driven SGS modeling

Applying a filter G to the NS equation: $\widetilde{u} = G * u$

$$
\begin{aligned}
&\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad \frac{\partial u_i}{\partial x_i} = 0, \\
&\frac{\partial \widetilde{u}_i}{\partial t} + \widetilde{u}_j \frac{\partial \widetilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \widetilde{p}}{\partial x_i} + \nu \frac{\partial^2 \widetilde{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}, \quad \frac{\partial \widetilde{u}_i}{\partial x_i} = 0,
\end{aligned} \tag{1}
$$

closure term: $\quad \tau_{ij} = \widetilde{u_i u_j} - \widetilde{u}_i \widetilde{u}_j.$

Classical modeling: empirical models, i.e. Smagorinsky model;

Data-driven SGS modeling: solve a regression problem:

$$
\min_{\theta} \sum_n \left\| \phi_\theta(\widetilde{\mathbf{u}}^{(n)}) - \tau^{(n)} \right\|^2 . \tag{2}
$$

generative SGS model

Observation on the SGS modeling dataset:

Data imbalance: a large potion of data has negligible magnitude.

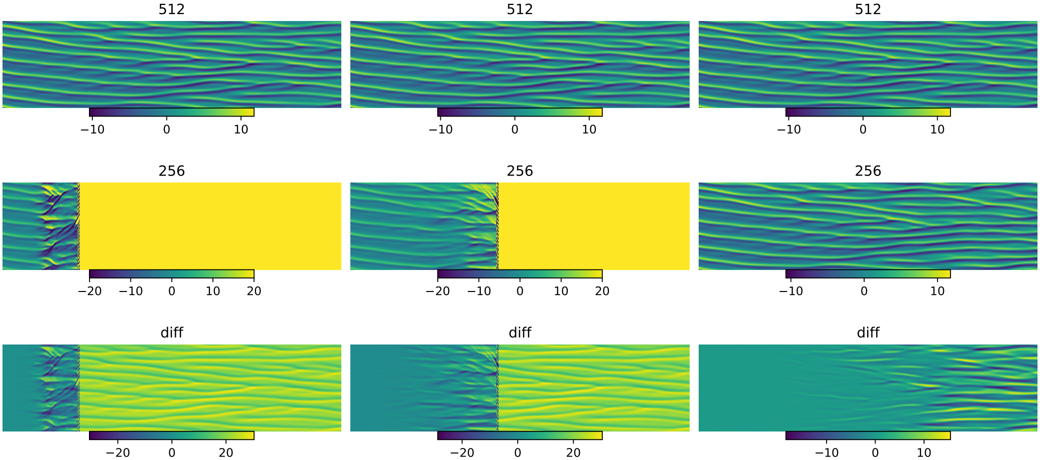
Multivaluedness: multiple values of stresses correspond to a single input, resulting in a large training error.

Solution to multivaluedness: modeling the closure term as a distribution conditioned on the input

$$\tau = \phi_\theta(u) \quad \rightarrow \quad \tau \sim p_\theta(\cdot|u).$$

$$\max_\theta \sum_{i=n}^{N} \log p_\theta(\tau^{(n)}|u^{(n)}) \Longleftrightarrow \min_\theta \sum_{n=1}^{N} \frac{(\tau^{(n)} - \mu_\theta(u^{(n)}))^2}{2(\sigma_\theta(u^{(n)}))^2} + \log \sigma_\theta(u^{(n)}).$$

Comparison with NN ansatz and Smagorinsky model ansatz



(a) Regression ansatz: neural network

(b) Regression ansatz: Smagorinsky

(c) Probabilistic ansatz: Gaussian